

# FUNK, COSINE, AND SINE TRANSFORMS ON STIEFEL AND GRASSMANN MANIFOLDS, II

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**ABSTRACT.** We investigate analytic continuation of the matrix cosine and sine transforms introduced in Part I and depending on a complex parameter  $\alpha$ . It is shown that the cosine transform corresponding to  $\alpha = 0$  is a constant multiple of the Funk-Radon transform in integral geometry for a pair of Stiefel (or Grassmann) manifolds. The same case for the sine transform gives the identity operator. These results and the relevant composition formula for the cosine transforms were established in Part I in the sense of distributions. Now we have them pointwise. Some new problems are formulated.

## 1. INTRODUCTION

Let  $V_{n,m}$  and  $V_{n,k}$  be a pair of Stiefel manifolds of orthonormal frames in  $\mathbb{R}^n$  of dimensions  $m$  and  $k$ , respectively;  $1 \leq m, k \leq n-1$ . The following integral operators were introduced in [34]:

$$(\mathcal{C}_{m,k}^\alpha f)(u) = \int_{V_{n,m}} f(v) [\det(v'uu'v)]^{(\alpha-k)/2} d_*v, \quad (1.1)$$

$$(\mathcal{C}_{m,k}^{\alpha*} \varphi)(v) = \int_{V_{n,k}} \varphi(u) [\det(v'uu'v)]^{(\alpha-k)/2} d_*u, \quad (1.2)$$

$$(\mathcal{S}_{m,k}^\alpha f)(u) = \int_{V_{n,m}} f(v) [\det(I_m - v'uu'v)]^{(\alpha+k-n)/2} d_*v, \quad (1.3)$$

$$(\mathcal{S}_{m,k}^{\alpha*} \varphi)(v) = \int_{V_{n,k}} \varphi(u) [\det(I_m - v'uu'v)]^{(\alpha+k-n)/2} d_*u. \quad (1.4)$$

Here  $u \in V_{n,k}$ ,  $v \in V_{n,m}$ ,  $d_*u$  and  $d_*v$  stand for the relevant probability measures,  $(\cdot)'$  denotes the transpose of a matrix  $(\cdot)$ .

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We call  $\mathcal{C}_{m,k}^\alpha f$  and  $\mathcal{S}_{m,k}^\alpha f$  the *cosine transform* and the *sine transform* of  $f$ , respectively. Integrals  $\mathcal{C}_{m,k}^{\alpha*} \varphi$  and  $\mathcal{S}_{m,k}^{\alpha*} \varphi$  are called the *dual cosine transform* and the *dual sine transform*. The terminology stems from the fact that, in the case  $k = m = 1$ , when  $u$  and  $v$  are unit vectors,

$$\det(v'uu'v) = (u \cdot v)^2 = \cos^2 \omega, \quad \det(I_m - v'uu'v) = 1 - (u \cdot v)^2 = \sin^2 \omega,$$

where  $\omega$  is the angle between  $u$  and  $v$ . For integrable functions  $f$  and  $\varphi$ , integrals (1.1)-(1.4) converge absolutely almost everywhere if and only if  $\operatorname{Re} \alpha > m - 1$  and represent analytic functions of  $\alpha$  in this domain; see [34, Theorem 4.2].

The present article is a continuation of our previous work [34] devoted to the study of operators (1.1)-(1.4). The impetus for this research was given by the celebrated Matheron's conjecture in stochastic geometry [22, p. 189] and a series of related publications in the area of harmonic analysis and integral geometry due to Goodey and Howard [14], Alesker and Bernstein [2], Alesker [1], Ournycheva and Rubin [26, 27], Zhang [41]; see [34] for the detailed exposition of the history and motivation of this research.

In the present paper we suggest an elementary approach to analytic continuation of the cosine transform, which enables us to study analytic properties of the sine transform and the duals of both transforms. We answer a series of questions stated in [34] and related to pointwise equalities, the validity of which was proved in Part I only in the sense of distributions.

In section 2 we recall basic definitions and auxiliary facts, which are used in the proofs. Main results are presented by Theorems 3.1, 4.1, 5.1 in sections 3, 4 and 5, respectively. Section 6 contains some consequences, including pointwise inversion of the relevant Funk-Radon transform and a composition formula (Theorems 6.1 and 6.4). We conclude with brief discussion of the so-called  $\operatorname{Cos}^\lambda$ -transform, which represents a particular case  $k = m$  of (1.1) and differs from the latter by notation. Such transforms arise in group representations [4, 24, 28].

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## 2. PRELIMINARIES

More information about facts presented below can be found in [16, 25, 33, 34].

**2.1. Notation and conventions.** Given a square matrix  $a$ ,  $|a|$  stands for the absolute value of  $\det(a)$ . As usual,  $O(n)$  and  $SO(n)$  stand for

the orthogonal group and the special orthogonal group of  $\mathbb{R}^n$ , respectively, with the normalized invariant measure of total mass 1. The abbreviation “a.c.” denotes analytic continuation.

Let  $\mathfrak{M}_{n,m} \sim \mathbb{R}^{nm}$  be the space of real matrices  $x = (x_{i,j})$  having  $n$  rows and  $m$  columns;  $dx = \prod_{i=1}^n \prod_{j=1}^m dx_{i,j}$ ;  $x'$  is the transpose of  $x$ ,  $|x|_m = \det(x'x)^{1/2}$ ,  $I_m$  is the identity  $m \times m$  matrix, and 0 stands for zero entries.

Let  $\mathcal{S}_m \sim \mathbb{R}^{m(m+1)/2}$  be the space of  $m \times m$  real symmetric matrices  $r = (r_{i,j})$ ;  $dr = \prod_{i \leq j} dr_{i,j}$ . We denote by  $\Omega$  the cone of positive definite matrices in  $\mathcal{S}_m$ . The Siegel gamma function of  $\Omega$  is defined by

$$\Gamma_m(\alpha) = \int_{\Omega} \exp(-\text{tr}(r)) |r|^{\alpha-(m+1)/2} dr = \pi^{m(m-1)/4} \prod_{j=0}^{m-1} \Gamma(\alpha-j/2). \quad (2.1)$$

This integral is absolutely convergent if and only if  $\text{Re } \alpha > (m-1)/2$ , and extends meromorphically with the polar set

$$\{(m-1-j)/2 : j = 0, 1, 2, \dots\}; \quad (2.2)$$

see [10], [5], [39].

For  $n \geq m$ , let  $V_{n,m} = \{v \in \mathfrak{M}_{n,m} : v'v = I_m\}$  be the Stiefel manifold of orthonormal  $m$ -frames in  $\mathbb{R}^n$ . This is a homogeneous space with respect to the action  $V_{n,m} \ni v \rightarrow \gamma v$ ,  $\gamma \in O(n)$ , so that  $V_{n,m} = O(n)/O(n-m)$ . We fix a measure  $dv$  on  $V_{n,m}$ , which is left  $O(n)$ -invariant, right  $O(m)$ -invariant, and normalized by

$$\sigma_{n,m} \equiv \int_{V_{n,m}} dv = \frac{2^m \pi^{nm/2}}{\Gamma_m(n/2)}, \quad (2.3)$$

[23, p. 70]. The notation  $d_*v = \sigma_{n,m}^{-1} dv$  is used for the corresponding probability measure.

We denote by  $G_{n,m}$  the Grassmann manifold of  $m$ -dimensional linear subspaces  $\xi$  of  $\mathbb{R}^n$  equipped with the  $O(n)$ -invariant probability measure  $d_*\xi$ . Every right  $O(m)$ -invariant function  $f(v)$  on  $V_{n,m}$  can be identified with a function  $\tilde{f}(\xi)$  by the formula  $\tilde{f}(\{v\}) = f(v)$ ,  $\{v\} = v\mathbb{R}^m \in G_{n,m}$ , so that  $\int_{G_{n,m}} \tilde{f}(\xi) d_*\xi = \int_{V_{n,m}} f(v) d_*v$ . Another identification is also possible, namely,  $\tilde{f}(\{v\}^\perp) = f(v)$ ,  $\{v\}^\perp \in G_{n,n-m}$ .

**Lemma 2.1.** (The polar decomposition). *Let  $x \in \mathfrak{M}_{n,m}$ ,  $n \geq m$ . If  $\text{rank}(x) = m$ , then*

$$x = vr^{1/2}, \quad v \in V_{n,m}, \quad r = x'x \in \Omega, \quad (2.4)$$

and  $dx = 2^{-m} |r|^{(n-m-1)/2} dr dv$ .

For this statement see, e.g., [18], [23], [5].

**2.2. Zeta integrals.** Let  $S(\mathfrak{M}_{n,m})$  be the Schwartz space of infinitely differentiable functions on  $\mathfrak{M}_{n,m}$  which are rapidly decreasing together with derivatives of all orders. Suppose that  $n \geq m \geq 2$  and consider the zeta integral

$$\mathcal{Z}(f, \alpha - n) = \int_{\mathfrak{M}_{n,m}} f(x) |x|_m^{\alpha-n} dx, \quad f \in S(\mathfrak{M}_{n,m}). \quad (2.5)$$

**Lemma 2.2.** *The integral (2.5) is absolutely convergent if  $\operatorname{Re} \alpha > m-1$  and extends to  $\operatorname{Re} \alpha \leq m-1$  as a meromorphic function of  $\alpha$  with the only poles  $m-1, m-2, \dots$ . These poles and their orders are exactly the same as of the gamma function  $\Gamma_m(\alpha/2)$ . The normalized zeta integral  $\mathcal{Z}(f, \alpha - n)/\Gamma_m(\alpha/2)$  is an entire function of  $\alpha$ .*

This statement can be found in [36, 21]; see also [33, Lemma 4.2]. The *Cayley-Laplace operator*  $\Delta$  on the space  $\mathfrak{M}_{n,m}$  of matrices  $x = (x_{i,j})$  is defined by

$$\Delta = \det(\partial' \partial). \quad (2.6)$$

Here  $\partial$  is an  $n \times m$  matrix whose entries are partial derivatives  $\partial/\partial x_{i,j}$ . More information about this operator, which is neither elliptic nor hyperbolic, can be found in [21, 33]. The following identity of the Bernstein type holds:

$$\Delta^\ell |x|_m^{\alpha+2\ell-n} = B_\ell(\alpha) |x|_m^{\alpha-n}, \quad (2.7)$$

$$B_{\ell,m,n}(\alpha) = \prod_{i=0}^{m-1} \prod_{j=0}^{\ell-1} (\alpha - i + 2j)(\alpha - n + 2 + 2j + i); \quad (2.8)$$

see [33, p. 565]. It allows us to represent meromorphic continuation of  $\mathcal{Z}(f, \alpha - n)$  in the form

$$\mathcal{Z}(f, \alpha - n) = \frac{1}{B_\ell(\alpha)} \mathcal{Z}(\Delta^\ell f, \alpha + 2\ell - n), \quad \operatorname{Re} \alpha > m - 1 - 2\ell, \quad (2.9)$$

$$\ell = 1, 2, \dots, \infty.$$

**Lemma 2.3.** [33, Theorem 4.4] *For  $f \in \mathcal{S}(\mathfrak{M}_{n,m})$ ,*

$$\lim_{\alpha \rightarrow 0} \frac{\mathcal{Z}(f, \alpha - n)}{\Gamma_m(\alpha/2)} = \frac{\pi^{nm/2}}{\Gamma_m(n/2)} f(0). \quad (2.10)$$

The general references related to zeta integrals are fundamental works by Bopp and Rubenthaler [3], Igusa [19], Shintani [38].

**2.3. The Funk Transform.** The classical Funk transform on the unit sphere  $S^{n-1} \subset \mathbb{R}^n$  is defined by

$$(Ff)(u) = \int_{\{v \in S^{n-1} : u \cdot v = 0\}} f(v) d_u v, \quad u \in S^{n-1}; \quad (2.11)$$

see, e.g., [7, 17]. In [34] we introduced the following generalization of (2.11), in which  $u \in V_{n,k}$  and  $v \in V_{n,m}$  are elements of the respective Stiefel manifolds,  $1 \leq k, m \leq n-1$ . The *higher-rank Funk transform* sends a function  $f$  on  $V_{n,m}$  to a function  $F_{m,k}f$  on  $V_{n,k}$  by the formula

$$(F_{m,k}f)(u) = \int_{\{v \in V_{n,m} : u'v = 0\}} f(v) d_u v, \quad u \in V_{n,k}. \quad (2.12)$$

The corresponding dual transform

$$(\bar{F}_{m,k}\varphi)(v) = \int_{\{u \in V_{n,k} : v'u = 0\}} \varphi(u) d_v u, \quad v \in V_{n,m}, \quad (2.13)$$

acts in the opposite direction. The condition  $u'v = 0$  means that subspaces  $u\mathbb{R}^k \in G_{n,k}$  and  $v\mathbb{R}^m \in G_{n,m}$  are mutually orthogonal. Hence, necessarily,

$$k + m \leq n.$$

In the case  $k = m$  we denote  $F_m = F_{m,m}$ .

To give our transforms precise meaning, we set  $G = O(n)$ ,

$$K_0 = \left\{ \tau \in G : \tau = \begin{bmatrix} \gamma & 0 \\ 0 & I_k \end{bmatrix}, \quad \gamma \in O(n-k) \right\}, \quad (2.14)$$

$$\check{K}_0 = \left\{ \rho \in G : \rho = \begin{bmatrix} \delta & 0 \\ 0 & I_m \end{bmatrix}, \quad \delta \in O(n-m) \right\}, \quad (2.15)$$

$$u_0 = \begin{bmatrix} 0 \\ I_k \end{bmatrix}, \quad \check{u}_0 = \begin{bmatrix} I_k \\ 0 \end{bmatrix}; \quad v_0 = \begin{bmatrix} 0 \\ I_m \end{bmatrix}, \quad \check{v}_0 = \begin{bmatrix} I_m \\ 0 \end{bmatrix}, \quad (2.16)$$

$$u_0, \check{u}_0 \in V_{n,k}; \quad v_0, \check{v}_0 \in V_{n,m}.$$

Then (2.12) and (2.13) can be explicitly written as

$$(F_{m,k}f)(u) = \int_{V_{n-k,m}} f\left(g_u \begin{bmatrix} \omega \\ 0 \end{bmatrix}\right) d_*\omega = \int_{K_0} f(g_u \tau \check{v}_0) d\tau, \quad (2.17)$$

$$(\bar{F}_{m,k}\varphi)(v) = \int_{V_{n-m,k}} \varphi\left(g_v \begin{bmatrix} \theta \\ 0 \end{bmatrix}\right) d_*\theta = \int_{\check{K}_0} \varphi(g_v \rho \check{u}_0) d\rho, \quad (2.18)$$

where  $g_u$  and  $g_v$  are orthogonal transformations satisfying  $g_u u_0 = u$  and  $g_v v_0 = v$ , respectively.

**Lemma 2.4.** [34, Lemma 3.2] *Let  $1 \leq k, m \leq n-1$ ;  $k+m \leq n$ . Then*

$$\int_{V_{n,k}} (F_{m,k}f)(u) \varphi(u) d_*u = \int_{V_{n,m}} f(v) (F_{m,k}^*\varphi)(v) d_*v \quad (2.19)$$

*provided that at least one of these integrals is finite when  $f$  and  $\varphi$  are replaced by  $|f|$  and  $|\varphi|$ , respectively.*

**2.4. Cosine and Sine transforms.** When dealing with operators (1.1) and (1.2), we restrict our consideration to the case  $m \leq k$ , because, if  $m > k$ , then  $|v'uu'v| = 0$  for all  $v \in V_{n,m}$  and all  $u \in V_{n,k}$ . Similarly, for (1.3) and (1.4), we assume  $m \leq n-k$ , because, if  $m > n-k$ , then

$$|I_m - v'uu'v| = |I_m - v'\text{Pr}_{\{u\}}v| = |v'\text{Pr}_{\{u\}^\perp}v| = |v'\tilde{u}\tilde{u}'v| = 0 \quad (2.20)$$

(here  $\tilde{u}$  is an arbitrary  $(n-k)$ -frame orthogonal to  $\{u\} = u\mathbb{R}^k$ ). The case  $k = n$ , when  $v'uu'v \equiv I_m$ , is also not interesting. Clearly,

$$(\mathcal{S}_{m,k}^\alpha f)(u) = (\mathcal{C}_{m,n-k}^\alpha f)(\tilde{u}) = \int_{V_{n,m}} f(v) |v'\tilde{u}\tilde{u}'v|^{(\alpha-(n-k))/2} d_*v. \quad (2.21)$$

The case  $k = m$ , when  $\mathcal{C}_{m,k}^\alpha$  and  $\mathcal{S}_{m,k}^\alpha$  coincide with their duals, is of particular importance. In this case we denote

$$(M^\alpha f)(u) = \int_{V_{n,m}} f(v) |u'v|^{\alpha-m} d_*v, \quad 1 \leq m \leq n-1, \quad (2.22)$$

$$(Q^\alpha f)(u) = \int_{V_{n,m}} f(v) |I_m - v'uu'v|^{(\alpha+m-n)/2} d_*v, \quad 2m \leq n, \quad (2.23)$$

where  $u \in V_{n,m}$ .

### 3. ANALYTIC CONTINUATION OF THE COSINE TRANSFORM

We use the following indirect procedure. Consider the cosine transform

$$(\mathcal{C}_{m,k}^\alpha f)(u) = \int_{V_{n,m}} f(v) |v'uu'v|^{(\alpha-k)/2} d_*v, \quad u \in V_{n,k}, \quad (3.1)$$

and set  $u = \gamma u_0$ ,  $u_0 = \begin{bmatrix} 0 \\ I_k \end{bmatrix} \in V_{n,k}$ ,  $\gamma \in SO(n)$ . Changing variable  $v = \gamma w$ , we obtain

$$(\mathcal{C}_{m,k}^\alpha f)(\gamma u_0) = \int_{V_{n,m}} f_\gamma(w) |w'u_0u_0'w|^{(\alpha-k)/2} d_*w,$$

where  $f_\gamma(w) = f(\gamma w)$ . Consider an auxiliary integral

$$F_\alpha(\gamma) = \int_{\mathfrak{M}_{n,m}} f_\gamma(x(x'x)^{-1/2}) |x'u_0u_0'x|^{(\alpha-k)/2} \psi(x'x) \exp(-\text{tr}(x'x)) dx,$$

where  $\psi(r)$  is a nonnegative  $C^\infty$  function on the cone  $\Omega$  having a compact support away from the boundary of  $\Omega$ . Suppose that  $f$  belongs to  $C^\infty(V_{n,m})$ . Then the function

$$\varphi_\gamma(x) \equiv f_\gamma(x(x'x)^{-1/2}) \psi(x'x) \exp(-\text{tr}(x'x)) \quad (3.2)$$

belongs to  $S(\mathfrak{M}_{n,m})$  and is supported away from the surface  $|x'x| = 0$ . Passing to polar coordinates  $x = wr^{1/2}$ ,  $w \in V_{n,m}$ ,  $r \in \Omega$  (cf. (2.4)), we obtain

$$F_\alpha(\gamma) = \varkappa(\alpha) (\mathcal{C}_{m,k}^\alpha f)(\gamma u_0), \quad (3.3)$$

$$\varkappa(\alpha) = 2^{-m} \sigma_{n,m} \int_{\Omega} |r|^{(\alpha-k+n)/2-d} \psi(r) \exp(-\text{tr}(r)) dr, \quad (3.4)$$

$d = (m+1)/2$ . Since  $\varkappa(\alpha)$  and its reciprocal are entire functions of  $\alpha$ , which differ from zero for any complex  $\alpha$ , then analyticity of  $\mathcal{C}_{m,k}^\alpha f$  is equivalent to that of  $F_\alpha(\gamma)$  and possible poles of both functions are at the same points and of the same orders.

The function  $F_\alpha(\gamma)$  can be represented as a zeta integral, namely,

$$\begin{aligned} F_\alpha(\gamma) &= \int_{\mathfrak{M}_{n,m}} \varphi_\gamma(x) |x' u_0 u_0' x|^{(\alpha-k)/2} dx \\ &= \int_{\mathfrak{M}_{k,m}} |y|_m^{\alpha-k} \tilde{\varphi}_\gamma(y) dy = \mathcal{Z}(\tilde{\varphi}_\gamma, \alpha - k), \end{aligned} \quad (3.5)$$

where the function

$$\tilde{\varphi}_\gamma(y) = \int_{\mathfrak{M}_{n-k,m}} \varphi_\gamma \left( \begin{bmatrix} \eta \\ y \end{bmatrix} \right) d\eta \quad (3.6)$$

belongs to  $S(\mathfrak{M}_{k,m})$ ; cf. (2.5). Thus,

$$(\mathcal{C}_{m,k}^\alpha f)(\gamma u_0) = \varkappa(\alpha)^{-1} \mathcal{Z}(\tilde{\varphi}_\gamma, \alpha - k). \quad (3.7)$$

**Theorem 3.1.** *Let  $1 \leq m \leq k \leq n-1$ ,  $f \in C^\infty(V_{n,m})$ .*

- (i) *If  $\text{Re } \alpha > m-1$ , then the cosine transform  $(\mathcal{C}_{m,k}^\alpha f)(u)$  is represented by an absolutely convergent integral.*
- (ii) *For every  $u \in V_{n,k}$ , the function  $\alpha \rightarrow (\mathcal{C}_{m,k}^\alpha f)(u)$  extends to the domain  $\text{Re } \alpha \leq m-1$  as a meromorphic function with the only poles  $m-1, m-2, \dots$ . These poles and their orders are exactly the same as of the gamma function  $\Gamma_m(\alpha/2)$ .*
- (iii) *The meromorphic continuation of  $(\mathcal{C}_{m,k}^\alpha f)(u)$  can be represented in the form*

$$(\mathcal{C}_{m,k}^\alpha f)(\gamma u_0) = \frac{1}{\varkappa(\alpha) B_{\ell,m,k}(\alpha)} \mathcal{Z}(\Delta^\ell \tilde{\varphi}_\gamma, \alpha + 2\ell - k), \quad (3.8)$$

$$\text{Re } \alpha > m-1-2\ell, \quad \ell = 1, 2, \dots,$$

where  $B_{\ell,m,k}(\alpha)$  is the Bernstein polynomial

$$B_{\ell,m,k}(\alpha) = \prod_{i=0}^{m-1} \prod_{j=0}^{\ell-1} (\alpha - i + 2j)(\alpha - k + 2 + 2j + i) \quad (3.9)$$

and  $\tilde{\varphi}_\gamma$  is defined by (3.6)-(3.2).

(iv) The normalized integral  $(\mathcal{C}_{m,k}^\alpha f)(u)/\Gamma_m(\alpha/2)$  is an entire function of  $\alpha$  and belongs to  $C^\infty(V_{n,k})$  in the  $u$ -variable.

(v) If, moreover,  $k + m \leq n$ , then

$$\text{a.c.}_{\alpha=0} \frac{(\mathcal{C}_{m,k}^\alpha f)(u)}{\Gamma_m(\alpha/2)} = c_{k,m} (F_{m,k} f)(u), \quad (3.10)$$

where

$$c_{k,m} = \frac{\Gamma_m(n/2)}{\Gamma_m(k/2) \Gamma_m((n-k)/2)},$$

and  $(F_{m,k} f)(u)$  is the Funk transform (2.17).

*Proof.* Statements (i) - (iv) are immediate consequences of Lemma 2.2 and equality (2.9). In particular, smoothness of the analytic continuation

$$\text{a.c.} (\mathcal{C}_{m,k}^\alpha f)(u)/\Gamma_m(\alpha/2) \quad (3.11)$$

in the  $u$ -variable follows from (3.8), because  $\Delta^\ell \tilde{\varphi}_\gamma$  is a smooth function of  $\gamma \in SO(n)$ . To prove (v), by (3.7) we have

$$\text{a.c.}_{\alpha=0} \frac{(\mathcal{C}_{m,k}^\alpha f)(u)}{\Gamma_m(\alpha/2)} = \text{a.c.}_{\alpha=0} \frac{1}{\varkappa(\alpha)} \frac{\mathcal{Z}(\tilde{\varphi}_\gamma, \alpha - k)}{\Gamma_m(\alpha/2)}. \quad (3.12)$$

Hence, by (2.10), (3.6), and (3.2),

$$\begin{aligned} \text{a.c.}_{\alpha=0} \frac{(\mathcal{C}_{m,k}^\alpha f)(u)}{\Gamma_m(\alpha/2)} &= \frac{\pi^{km/2}}{\varkappa(0) \Gamma_m(k/2)} \tilde{\varphi}_\gamma(0) \\ &= \frac{\pi^{km/2}}{\varkappa(0) \Gamma_m(k/2)} \int_{\mathfrak{M}_{n-k,m}} f_\gamma \left( \begin{bmatrix} \eta \\ 0 \end{bmatrix} (\eta' \eta)^{-1/2} \right) \psi(\eta' \eta) \exp(-\text{tr}(\eta' \eta)) d\eta. \end{aligned}$$

Since  $n - k \geq m$ , we can use Lemma 2.1 to pass to polar coordinates and get an expression of the form  $c I_1 I_2$ , where

$$I_1 = \int_{V_{n-k,m}} f_\gamma \left( \begin{bmatrix} \omega \\ 0 \end{bmatrix} \right) d_* \omega, \quad I_2 = \int_{\Omega} |r|^{(n-k)/2-d} \psi(r) \exp(-\text{tr}(r)) dr,$$

$$c = \frac{\pi^{km/2} \sigma_{n-k,m}}{\sigma_{n,m} \Gamma_m(k/2) I_2} = \frac{\Gamma_m(n/2)}{\Gamma_m(k/2) \Gamma_m((n-k)/2)} \frac{1}{I_2}$$



(use (2.3)). Integral  $I_1$  is exactly the Funk transform (2.17). This proves (3.10).  $\square$

Some comments are in order.

1. Equality (3.10) was obtained in [34] in the weak sense, using the Fourier transform technique; cf. formula (6.9) in that paper. Our new proof is much simpler and more informative.
2. The particular case  $k = m$  in Theorem 3.1 characterizes analytic properties of the operator  $M^\alpha f$  defined by (2.22). For example, (3.10) yields

$$\text{a.c.}_{\alpha=0} \frac{(M^\alpha f)(u)}{\Gamma_m(\alpha/2)} = c_{m,m} (F_m f)(u), \quad (3.13)$$

where

$$c_{m,m} = \frac{\Gamma_m(n/2)}{\Gamma_m(m/2) \Gamma_m((n-m)/2)}, \quad 2m \leq n,$$

and

$$(F_m f)(u) = \int_{\{v \in V_{n,m} : u'v=0\}} f(v) d_u v, \quad u \in V_{n,m}, \quad (3.14)$$

is the Funk transform.

3. The particular case  $f \equiv 1$ , when

$$(\mathcal{C}_{m,k}^\alpha 1)(u) \equiv \int_{V_{n,m}} |v' u u' v|^{(\alpha-k)/2} dv = \frac{\Gamma_m(n/2)}{\Gamma_m(k/2)} \frac{\Gamma_m(\alpha/2)}{\Gamma_m((\alpha-k+n)/2)} \quad (3.15)$$

(cf. (8.16) in [34]) illustrates the role of the gamma function  $\Gamma_m(\alpha/2)$  in the statement (ii) of Theorem 3.1.

4. It is known [25, 33] that analytic continuation of the zeta integral (2.5) at the so-called *Wallah set*  $\alpha = 0, 1, 2, \dots, m-1$  is represented as a convolution with a positive measure, which can be explicitly evaluated. We conjecture, that a similar result holds for the normalized cosine transform  $(\mathcal{C}_{m,k}^\alpha f)(u)/\Gamma_m(\alpha/2)$  and the corresponding measure can be explicitly evaluated in the form, which does not contain the auxiliary function  $\psi$  (so far, we succeeded in doing that only for  $\alpha = 0$ ).

#### 4. ANALYTIC CONTINUATION OF THE SINE TRANSFORM

The sine transform of a function  $f(v)$  on  $V_{n,m}$  is a function  $(\mathcal{S}_{m,k}^\alpha f)(u)$  defined by

$$(\mathcal{S}_{m,k}^\alpha f)(u) = \int_{V_{n,m}} f(v) |I_m - v' u u' v|^{(\alpha+k-n)/2} d_* v, \quad u \in V_{n,k}. \quad (4.1)$$

We assume  $k + m \leq n$ , because otherwise  $|I_m - v' u u' v| = 0$  for all  $v \in V_{n,m}$  and  $u \in V_{n,k}$ ; see (2.20).

**Theorem 4.1.** *Let  $1 \leq k, m \leq n-1$ ,  $k+m \leq n$ ,  $f \in C^\infty(V_{n,m})$ .*

- (i) *If  $\operatorname{Re} \alpha > m-1$ , then the sine transform  $(\mathcal{S}_{m,k}^\alpha f)(u)$  is represented by an absolutely convergent integral.*
- (ii) *For every  $u \in V_{n,k}$ , the function  $\alpha \rightarrow (\mathcal{S}_{m,k}^\alpha f)(u)$  extends to the domain  $\operatorname{Re} \alpha \leq m-1$  as a meromorphic function with the only poles  $m-1, m-2, \dots$ . These poles and their orders are exactly the same as of the gamma function  $\Gamma_m(\alpha/2)$ .*
- (iii) *The normalized integral  $(\mathcal{S}_{m,k}^\alpha f)(u)/\Gamma_m(\alpha/2)$  is an entire function of  $\alpha$  and belongs to  $C^\infty(V_{n,k})$  in the  $u$ -variable.*
- (iv) *In the case  $k = m$ ,  $2m \leq n$ , when  $\mathcal{S}_{m,k}^\alpha f \equiv Q^\alpha f$  is the integral (2.23), we have*

$$\text{a.c.} \frac{(Q^\alpha f)(u)}{\Gamma_m(\alpha/2)} = \frac{\Gamma_m(n/2)}{\Gamma_m(m/2) \Gamma_m((n-m)/2)} f(u) \quad (4.2)$$

*provided that  $f$  is right  $O(m)$ -invariant.*

*Proof.* We have

$$|I_m - v' u u' v| = |I_m - v' \operatorname{Pr}_{\{u\}} v| = |v' \operatorname{Pr}_{\{u\}^\perp} v| = |v' \tilde{u} \tilde{u}' v| = 0,$$

where  $\tilde{u}$  is an arbitrary  $(n-k)$ -frame orthogonal to  $\{u\} = u\mathbb{R}^k$ . Hence,

$$(\mathcal{S}_{m,k}^\alpha f)(u) = \int_{V_{n,m}} f(v) |v' \tilde{u} \tilde{u}' v|^{(\alpha+k-n)/2} d_* v = (\mathcal{C}_{m,n-k}^\alpha f)(\tilde{u}). \quad (4.3)$$

It remains to apply Theorem 3.1. To prove (4.2), we make use of (2.21) with  $k = m$ . Then, as above,

$$\text{a.c.} \frac{(Q^\alpha f)(u)}{\Gamma_m(\alpha/2)} = \text{a.c.} \frac{(\mathcal{C}_{m,n-m}^\alpha f)(\tilde{u})}{\Gamma_m(\alpha/2)} = \tilde{c} \tilde{I}_1 \tilde{I}_2,$$

where

$$\begin{aligned} \tilde{I}_1 &= \int_{V_{m,m}} f_\gamma \left( \begin{bmatrix} \omega \\ 0 \end{bmatrix} \right) d_* \omega = \int_{O(m)} f \left( \gamma \begin{bmatrix} I_m \\ 0 \end{bmatrix} \omega \right) d\omega = f \left( \gamma \begin{bmatrix} I_m \\ 0 \end{bmatrix} \right), \\ I_2 &= \int_{\Omega} |r|^{m/2-d} \psi(r) \exp(-\operatorname{tr}(r)) dr, \quad \tilde{c} = \frac{\Gamma_m(n/2)}{\Gamma_m(m/2) \Gamma_m((n-m)/2)} \frac{1}{\tilde{I}_2}. \end{aligned}$$

Here  $\gamma$  is a rotation that takes  $\begin{bmatrix} 0 \\ I_{n-m} \end{bmatrix}$  to  $\tilde{u} \in \{u\}^\perp$ . Hence,  $\gamma$  takes

$\begin{bmatrix} I_m \\ 0 \end{bmatrix}$  to a certain frame, say,  $u_1$ , that spans the same subspace as  $u$ .

Since  $f$  is right  $O(m)$ -invariant, then  $\tilde{I}_1 = f(u)$ , and we are done.  $\square$

## 5. ANALYTIC CONTINUATION OF THE DUAL COSINE TRANSFORM

The dual cosine transform of a function  $\varphi$  on  $V_{n,k}$  is defined by

$$(\mathcal{C}_{m,k}^{\alpha} \varphi)(v) = \int_{V_{n,k}} \varphi(u) |v'uu'v|^{(\alpha-k)/2} d_* u, \quad v \in V_{n,m}, \quad m \leq k. \quad (5.1)$$

We will be dealing with smooth functions  $\varphi \in C^\infty(V_{n,k})$ . Moreover, since  $\varphi(u)$  and  $\varphi(u\gamma) \forall \gamma \in O(k)$  have the same dual cosine transform, we can assume that  $\varphi$  is right  $O(k)$ -invariant. We denote by  $\tilde{u} \in V_{n,n-k}$  and  $\tilde{v} \in V_{n,n-m}$  arbitrary frames orthogonal to subspaces  $\{u\}$  and  $\{v\}$ , respectively. Then

$$\begin{aligned} |v'uu'v| &= |I_m - v'\tilde{u}\tilde{u}'v| = |I_{n-k} - \tilde{u}'vv'\tilde{u}| = |I_{n-k} - \tilde{u}'\text{Pr}_{\{v\}}\tilde{u}| \\ &= |\tilde{u}'\text{Pr}_{\{v\}^\perp}\tilde{u}| = |\tilde{u}'\tilde{v}\tilde{v}'\tilde{u}|. \end{aligned}$$

Setting  $\varphi_1(\tilde{u}) = \varphi(u)$ , we obtain

$$\begin{aligned} (\mathcal{C}_{m,k}^{\alpha} \varphi)(v) &= \int_{V_{n,n-k}} \varphi_1(\tilde{u}) |\tilde{u}'\tilde{v}\tilde{v}'\tilde{u}|^{(\alpha-k)/2} d_* \tilde{u} \\ &= (\mathcal{C}_{n-k,n-m}^{\alpha+n-k-m} \varphi_1)(\tilde{v}), \quad \tilde{v} \in V_{n,n-m}. \end{aligned} \quad (5.2)$$

Thus, analytic properties of  $\mathcal{C}_{m,k}^{\alpha} \varphi$  can be derived from Theorem 3.1.

**Theorem 5.1.** *Let  $1 \leq m \leq k \leq n-1$  and let  $\varphi$  be a right  $O(k)$ -invariant function in  $C^\infty(V_{n,k})$ .*

- (i) *If  $\text{Re } \alpha > m-1$ , then dual cosine transform  $(\mathcal{C}_{m,k}^{\alpha} \varphi)(v)$  is represented by an absolutely convergent integral.*
- (ii) *For every  $v \in V_{n,m}$ , the function  $\alpha \rightarrow (\mathcal{C}_{m,k}^{\alpha} \varphi)(v)$  extends to the domain  $\text{Re } \alpha \leq m-1$  as a meromorphic function with the only poles  $m-1, m-2, \dots$ . These poles and their orders are exactly the same as of the gamma function  $\Gamma_{n-k}((\alpha+n-k-m)/2)$ .*
- (iii) *The normalized integral  $(\mathcal{C}_{m,k}^{\alpha} \varphi)(v)/\Gamma_{n-k}((\alpha+n-k-m)/2)$  is an entire function of  $\alpha$  and belongs to  $C^\infty(V_{n,m})$  in the  $v$ -variable.*
- (iv) *If  $k+m \leq n$ , then  $(\mathcal{C}_{m,k}^{\alpha} \varphi)(v)/\Gamma_m(\alpha/2)$  extends as a meromorphic function with the only possible poles  $-1, -2, \dots$ . Moreover, for every  $v \in V_{n,m}$ ,*

$$\text{a.c.} \lim_{\alpha=0} \frac{(\mathcal{C}_{m,k}^{\alpha} \varphi)(v)}{\Gamma_m(\alpha/2)} = c_{k,m} (F_{m,k}^* \varphi)(v), \quad (5.3)$$

where

$$c_{k,m} = \frac{\Gamma_m(n/2)}{\Gamma_m(k/2) \Gamma_m((n-k)/2)}$$

and  $(F_{m,k}^* \varphi)(v)$  is the dual Funk transform (2.13).

(v) If  $k + m > n$ , then for every  $v \in V_{n,m}$ ,

$$\underset{\alpha=k+m-n}{a.c.} \frac{(\mathcal{C}_{m,k}^* \varphi)(v)}{\Gamma_{n-k}((\alpha + n - k - m)/2)} = \tilde{c}_{k,m} (F_{n-k,n-m} \varphi_1)(\tilde{v}), \quad (5.4)$$

where

$$\tilde{c}_{k,m} = \frac{\Gamma_{n-k}(n/2)}{\Gamma_{n-k}(m/2) \Gamma_{n-k}((n-m)/2)}$$

and  $(F_{n-k,n-m} \varphi_1)(\tilde{v})$  is the relevant Funk transform; cf. (2.17).

*Proof.* Owing to (5.2), statements (i)-(iii) and (v) are immediate consequences of the respective statements in Theorem 3.1. To prove (iv), we observe that by (2.1),

$$\Gamma_{n-k}((\alpha + n - k - m)/2) = c(\alpha) \Gamma_m(\alpha/2), \quad (5.5)$$

where  $c(\alpha) = \pi^{(n-k-m)m/2} \Gamma_{n-k-m}((\alpha + n - k - m)/2)$  is a meromorphic function with the polar set  $\{-1, -2, \dots\}$ . Denote

$$A_\alpha(v) \equiv \frac{(\mathcal{C}_{m,k}^* \varphi)(v)}{\Gamma_m(\alpha/2)} = \frac{c(\alpha) (\mathcal{C}_{m,k}^* \varphi)(v)}{\Gamma_{n-k}((\alpha + n - k - m)/2)}, \quad \operatorname{Re} \alpha > m - 1.$$

By (iii), this function extends analytically to  $\operatorname{Re} \alpha > -1$  and the analytic continuation belongs to  $C^\infty(V_{n,m})$ . Now for any test function  $\omega \in C^\infty(V_{n,m})$ , owing to (3.10) and (2.19), we have

$$\begin{aligned} (\underset{\alpha=0}{a.c.} A_\alpha, \omega) &= \underset{\alpha=0}{a.c.} \left( \varphi, \frac{\mathcal{C}_{m,k}^\alpha \omega}{\Gamma_m(\alpha/2)} \right) = \left( \varphi, \underset{\alpha=0}{a.c.} \frac{\mathcal{C}_{m,k}^\alpha \omega}{\Gamma_m(\alpha/2)} \right) \\ &= c_{k,m} (\varphi, F_{m,k} \omega) = c_{k,m} (F_{m,k}^* \varphi, \omega), \end{aligned}$$

and (5.3) follows.  $\square$

## 6. SOME CONSEQUENCES

### 6.1. Inversion of the Funk transform.

**Theorem 6.1.** *Let  $\varphi = F_{m,k} f$ , where  $f$  is a  $C^\infty$  right  $O(m)$ -invariant function on  $V_{n,m}$ ,  $1 \leq m \leq k \leq n - m$ . Then, for every  $v \in V_{n,m}$ ,*

$$\underset{\alpha=k+m-n}{a.c.} \frac{(\mathcal{C}_{m,k}^* \varphi)(v)}{\Gamma_m(\alpha/2)} = c f(v), \quad c = \frac{\Gamma_m(n/2)}{\Gamma_m(k/2) \Gamma_m(m/2)}. \quad (6.1)$$

*Proof.* By [34, Theorem 4.3], if  $\operatorname{Re} \alpha > m - 1$ , then

$$\frac{(\mathcal{C}_{m,k}^* \varphi)(v)}{\Gamma_m(\alpha/2)} = \frac{\Gamma_m((n-m)/2)}{\Gamma_m(k/2)} \frac{(Q^{\alpha+n-k-m} f)(v)}{\Gamma_m((\alpha + n - k - m)/2)}, \quad (6.2)$$

where  $Q^{\alpha+n-k-m}f$  is the sine transform from Section 4. By Theorem 4.1, the right-hand side of (6.2) extends as an entire function of  $\alpha$ . Now, taking analytic continuation and using (4.2), we obtain the result.  $\square$

**Remark 6.2.** It is interesting to note that in general, the function  $\alpha \rightarrow (\mathcal{C}_{m,k}^{\alpha} \varphi)(v)/\Gamma_m(\alpha/2)$  admits poles  $-1, -2, \dots$ ; see Theorem 5.1 (iv). However, as (6.2) reveals, these poles disappear if  $\mathcal{C}_{m,k}^{\alpha}$  acts on the image of the Funk transform  $F_{m,k}$ .

**Remark 6.3.** Theorem 6.1 can be reformulated in the language of Grassmannians. The reader can easily do this, using connection formulas from [34, Section 3.2]. A series of inversion results for the Funk-Radon transform on Grassmannians can be found in the fundamental works by Gelfand and his collaborators [8, 9], Grinberg [15], Kakehi [20]; see also a pioneering paper by Petrov [29], Grinberg and Rubin [16], and Zhang [40]. Diverse problems of integral geometry related to Grassmann manifolds were studied in [6, 11, 12, 13, 17, 32, 37]. Our Theorem 6.1 has a completely different flavor, agrees with the known case  $m = 1$  (cf. formula (1.13) in [30]) and sheds new light to this circle of problems.

**6.2. A composition formula; the case  $k = m$ .** In this particular case the cosine transform  $(M^{\alpha}f)(u) = (\mathcal{C}_{m,m}^{\alpha}f)(u)$ ,  $u \in V_{n,m}$ , has a number of important features. We normalize it by setting

$$\begin{aligned} (\mathcal{M}^{\alpha}f)(u) &= \delta_{n,m}(\alpha) (M^{\alpha}f)(u) \\ &= \delta_{n,m}(\alpha) \int_{V_{n,m}} f(v) |u'v|^{\alpha-m} d_*v, \end{aligned} \quad (6.3)$$

$$\delta_{n,m}(\alpha) = \frac{\Gamma_m(m/2)}{\Gamma_m(n/2)} \frac{\Gamma_m((m-\alpha)/2)}{\Gamma_m(\alpha/2)}, \quad \alpha \notin \mathbb{N} = \{1, 2, \dots\}.$$

The excluded values of  $\alpha$  form the polar set of  $\Gamma_m((m-\alpha)/2)$ . Integral (6.3) converges absolutely for any integrable function  $f$  when  $\operatorname{Re} \alpha > m-1$ . If  $f \in C^{\infty}(V_{n,m})$ , then, by Theorem 3.1, analytic continuation of a function  $\alpha \rightarrow (\mathcal{M}^{\alpha}f)(u)$  is well-defined for all complex  $\alpha \neq 1, 2, \dots$  and belongs to  $C^{\infty}(V_{n,m})$ . We denote

$$(\mathcal{M}_{a.c.}^{\alpha}f)(u) = a.c. \lim_{\alpha \notin \mathbb{N}} (\mathcal{M}^{\alpha}f)(u). \quad (6.4)$$

**Theorem 6.4.** *Let  $f \in C^{\infty}(V_{n,m})$  be a right  $O(m)$ -invariant function on  $V_{n,m}$ ,  $2m \leq n$ . If  $\alpha, 2m - \alpha - n \notin \mathbb{N}$ , then for every  $u \in V_{n,m}$ ,*

$$(\mathcal{M}_{a.c.}^{2m-\alpha-n} \mathcal{M}_{a.c.}^{\alpha}f)(u) = f(u). \quad (6.5)$$

*Proof.* In [34, Theorem 6.4] we proved that if  $Re\alpha > m - 1$ ,  $\alpha \neq m, m + 1, m + 2, \dots$ , then

$$\underset{\beta=2m-\alpha-n}{a.c.} (\mathcal{M}^\beta \mathcal{M}^\alpha f, \omega) = (f, \omega), \quad (6.6)$$

for any test function  $\omega \in C^\infty(V_{n,m})$ . Since  $\mathcal{M}^\alpha f \in C^\infty(V_{n,m})$ , then, by Theorem 3.1, analytic continuation of a function  $\beta \rightarrow (\mathcal{M}^\beta \mathcal{M}^\alpha f)(u)$  is well-defined for all  $\beta \neq 1, 2, \dots$  and represents a  $C^\infty$  function on  $V_{n,m}$ . Hence,

$$(\mathcal{M}_{a.c.}^\beta \mathcal{M}^\alpha f, \omega) = (a.c. \mathcal{M}^\beta \mathcal{M}^\alpha f, \omega) = a.c.(\mathcal{M}^\beta \mathcal{M}^\alpha f, \omega),$$

and the aforementioned result from [34] yields  $(\mathcal{M}_{a.c.}^{2m-\alpha-n} \mathcal{M}^\alpha f, \omega) = (f, \omega)$ . Since, by Theorem 3.1, the function  $(\mathcal{M}_{a.c.}^\beta \mathcal{M}^\alpha f)(u)$  is smooth, then we have a pointwise equality

$$(\mathcal{M}_{a.c.}^{2m-\alpha-n} \mathcal{M}^\alpha f)(u) = f(u), \quad (6.7)$$

provided that

$$Re\alpha > m - 1, \quad \alpha \neq m, m + 1, m + 2, \dots$$

To extend this result to all complex  $\alpha$  such that  $\alpha, 2m - \alpha - n \notin \mathbb{N}$ , we observe that  $(\mathcal{M}_{a.c.}^{2m-\alpha-n} \mathcal{M}^\alpha f)(u)$  is separately analytic in  $\alpha$  and  $\beta$  in the domain  $\Lambda = \{(\alpha, \beta) \in \mathbb{C}^2 : \alpha \notin \mathbb{N}, \beta \notin \mathbb{N}\}$ . Hence, by the fundamental Hartogs theorem [35], this function is analytic in  $z = (\alpha, \beta) \in \Lambda$ . Now (6.5) follows from (6.7) by the uniqueness of analytic continuation.  $\square$

**6.3. The  $\text{Cos}^\lambda$ -transforms.** Transformation (2.22) can be found in the literature under different names (or without naming) and with different notation. For instance, let

$$(\text{Cos}^\lambda f)(u) = \int_{V_{n,m}} f(v) |u'v|^{\lambda-\rho} d_* v, \quad (6.8)$$

$$Re\lambda > \rho - 1, \quad \rho = n/2, \quad u \in V_{n,m};$$

cf. [24], where a similar operator is presented in a slightly different form. Following [24], we call (6.8) the  $\text{Cos}^\lambda$ -transform of  $f$ . The connection between (6.8) and (2.22) is

$$\text{Cos}^\lambda f = M^{\lambda+m-\rho} f. \quad (6.9)$$

In terms of (6.8) formula (3.13) becomes

$$\underset{\lambda=\rho-m}{a.c.} \frac{\text{Cos}^\lambda f}{\Gamma_m((\lambda + m - \rho)/2)} = c_{m,m} F_m f. \quad (6.10)$$

The composition formula (6.5) transforms to

$$\text{Cos}^{-\lambda} \text{Cos}^\lambda f = f, \quad \pm\lambda + m - \rho \neq 1, 2, \dots, \quad (6.11)$$

where  $\text{Cos}^\lambda$  denotes analytic continuation of the normalized integral

$$(\text{Cos}^\lambda f)(u) = \tilde{\delta}_{n,m}(\lambda) \int_{V_{n,m}} f(v) |u'v|^{\lambda-\rho} d_* v. \quad (6.12)$$

$$\tilde{\delta}_{n,m}(\lambda) = \frac{\Gamma_m(m/2)}{\Gamma_m(\rho)} \frac{\Gamma_m((\rho-\lambda)/2)}{\Gamma_m((\lambda+m-\rho)/2)}.$$

6.3.1. *The case  $m = 1$ .* This case deserves special mentioning. The corresponding cosine transforms on the unit sphere  $S^{n-1}$  in  $\mathbb{R}^n$  are well-known in analysis. We recall some results; see a survey article [31] and references therein. In the “ $\lambda$ -notation” the normalized “rank-one”  $\text{Cos}^\lambda$ -transform is defined as analytic continuation of the integral

$$(\text{Cos}^\lambda f)(u) = \frac{\pi^{1/2} \Gamma((\rho-\lambda)/2)}{\Gamma(\rho) \Gamma((\lambda-\rho+1)/2)} \int_{S^{n-1}} f(v) |u \cdot v|^{\lambda-\rho} d_* v, \quad (6.13)$$

where  $\rho = n/2$  and  $d_* v$  is the probability measure on  $S^{n-1}$ . Let  $\{Y_{j,\nu}(v)\}$  be an orthonormal basis of spherical harmonics on  $S^{n-1}$ . Here  $j = 0, 1, 2, \dots$ , and  $\nu = 1, 2, \dots, d_n(j)$ , where  $d_n(j)$  is the dimension of the subspace of spherical harmonics of degree  $j$ . If

$$f = \sum_{j,k} f_{j,k} Y_{j,k}, \quad f_{j,k} = \int_{S^{n-1}} f(v) Y_{j,k}(v) dv, \quad (6.14)$$

(the Fourier-Laplace expansion of  $f$ ), then

$$\text{Cos}^\lambda f = \sum_{j,k} c_{j,\lambda} f_{j,k} Y_{j,k}, \quad (6.15)$$

where

$$c_{j,\lambda} = \begin{cases} (-1)^{j/2} \frac{\Gamma((j+\rho-\lambda)/2)}{\Gamma((j+\rho+\lambda)/2)} & \text{if } j \text{ is even,} \\ 0 & \text{if } j \text{ is odd,} \end{cases} \quad (6.16)$$

Let  $L_{p,e} \equiv L_{p,e}(S^{n-1})$  be the Lebesgue space of even  $p$ -integrable functions on  $S^{n-1}$ . The asymptotic of  $c_{j,\lambda}$  as  $j \rightarrow \infty$  implies the following embeddings of the range  $\text{Cos}^\lambda(L_{p,e})$ ,  $1 < p < \infty$ , into the relevant Sobolev spaces  $L_{p,e}^\gamma$ .

**Theorem 6.5.** (cf. Corollary 3.3 in [31]) *Let*

$$\gamma_\pm = \text{Re } \lambda \pm \left| \frac{1}{p} - \frac{1}{2} \right| (n-1),$$

$$\lambda \notin \{\rho, \rho+2, \rho+4, \dots\} \cup \{-\rho-1, -\rho-3, -\rho-5, \dots\}.$$

*The following proper embeddings hold:*

$$L_{p,e}^{\gamma_-} \subset \text{Cos}^\lambda(L_{p,e}) \subset L_{p,e}^{\gamma_+}. \quad (6.17)$$

If  $p = 2$ , then

$$\text{Cos}^\lambda(L_{2,e}) = L_{2,e}^{\text{Re}\lambda}.$$

It is a challenging problem to possibly extend these results to the higher-rank case  $m > 1$ .

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